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LETTER TO THE EDITOR

A geometric description of Dirac monopoles†

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Abstract. Dirac magnetic monopoles are described in terms of principal U(1) bundles over the sphere S^2. The structure group U(1) is then extended to the group SU(2) and potentials are given in gauges free of string singularities. Resulting vector and scalar fields can be asymptotic quantities for non-Abelian monopoles.

It is known that a Dirac magnetic monopole does not admit electromagnetic potentials which are regular and single-valued outside the position of the pole (Dirac 1931, Wu and Yang 1975). If r, θ, ϕ denote spherical coordinates in the Minkowski space–time (with signature $+---$) adapted to the worldline of the monopole, then in the domains where $\theta \neq 0$ or $\theta \neq \pi$ the potential 1-form can be chosen as

$$\kappa(\cos\theta + 1)\,\mathrm{d}\phi \qquad (\theta \neq 0) \tag{1}$$

or

$$\kappa(\cos\theta - 1)\,\mathrm{d}\phi \qquad (\theta \neq \pi), \tag{2}$$

respectively. Forms (1) and (2) give rise to a single-valued, spherically symmetric magnetic field. The constant κ has the meaning of the total magnetic charge of the monopole and the Dirac quantisation condition says that κ is a multiple of the smallest non-vanishing charge, thus $\kappa = n\kappa_1$, where n is an integer (see Goddard and Olive (1978) for a review).

In terms of principal fibre bundles (Kobayashi and Nomizu 1963) the Dirac monopole with charge $n\kappa_1$ can be described by a connection ω_n on a U(1) bundle, denoted here by $L_n(S^2, \mathrm{U}(1))$, over the two-dimensional sphere in the physical three-dimensional space (Wu and Yang (1975), more recent references can be found in Quiros and Rodriguez (1983)). The angles θ, ϕ parametrise the sphere and the remaining coordinates r, t can be introduced by taking the product of L_n with R^2.

L_0 is the trivial bundle $S^2 \times \mathrm{U}(1)$ and ω_0 is the trivial connection, $\omega_0(x, a) = a^{-1}\,\mathrm{d}a$. In the following we assume $n \neq 0$ if not stated otherwise.

As it was noted by Trautman (1977) the bundle space L_n is the lens space $\mathrm{SU}(2)/Z_n$, where Z_n is the group of SU(2)-valued nth roots of identity matrix. If we denote an element of SU(2) by gZ_n, where $g \in \mathrm{SU}(2)$, then the action of U(1) on L_n is defined by

$$gZ_n \to (g\sqrt[n]{a})Z_n, \qquad a \in \mathrm{U}(1) \subset \mathrm{SU}(2) \tag{3}$$

where the group U(1) is identified with the set of diagonal matrices in SU(2), $a \leftrightarrow \mathrm{Diag}(\bar{a}, a)$.

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We choose $e_k = -i\sigma_k/2$ $(k = 1, 2, 3)$, σ_k being the Pauli matrices, as generators of the Lie algebra su(2). Then $[e_i, e_j] = \varepsilon_{ijk} e_k$ and e_k's constitute a basis orthonormal with respect to the Killing form divided by -2. This basis induces the left invariant basis θ^k of 1-forms on SU(2) such that the Maurer–Cartan form is $g^{-1}\,dg = e_k\theta^k$.

The common base space of L_n's can be identified with a two-dimensional sphere in su(2) in such a way that a direction described by the angles θ, ϕ corresponds to the vector

$$\sin\theta\cos\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\theta\, e_3. \tag{4}$$

With this convention we define the canonical projections $\pi_n: \mathrm{SU}(2)/Z_n \to S^2$ by $\pi_n(gZ_n) = \pi(g)$, where $\pi(g) = ge_3g^{-1}$. If

$$g(\phi, \theta, \chi) = \exp(\phi e_3)\exp(\theta e_2)\exp(\chi e_3), \qquad 0 \leqslant \phi < 2\pi, \qquad 0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \chi < 4\pi$$

is a parametrisation of SU(2) then $\pi(g(\phi, \theta, \chi))$ coincides with (4).

The connection form ω_n on L_n corresponding to the Dirac monopole of charge $n\kappa_1$ can be defined by

$$p_n^*\omega_n = ne_3\theta^3, \tag{5}$$

where p_n is the natural homomorphism of L_1 onto L_n, $p_n(g) = gZ_n$. The pullbacks of ω_n under the sections $p_n \circ \sigma_+$, $p_n \circ \sigma_-$, where $\sigma_\pm(\theta, \phi) = g(\phi, \theta, \pm\phi)$ and $\theta \neq 0$, $\theta \neq \pi$, respectively, yield the expressions

$$A_n^\pm = n(\cos\theta \pm 1)\, d\phi\, e_3, \tag{6}$$

which are equivalent to (1) and (2) for $\kappa = n\kappa_1$.

Further considerations are based on the existence of the mappings $\Lambda_n: \mathrm{SU}(2) \to \mathrm{SU}(2)$ such that $\Lambda_n(1) = 1$ and $\Lambda_n(ga) = \Lambda_n(g)a^n$ for any $a \in \mathrm{U}(1)$ and $g \in \mathrm{SU}(2)$. If

$$g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix},$$

where $z_1, z_2 \in \mathbb{C}$, $|z_1|^2 + |z_2|^2 = 1$, then

$$\Lambda_n(g) = (|z_1|^{2n} + |z_2|^{2n})^{-1/2} \begin{pmatrix} z_1^n & -\bar{z}_2^n \\ z_2^n & \bar{z}_1^n \end{pmatrix} \qquad \text{for } n \geqslant 1$$

and $\Lambda_n(g) = \Lambda_{-n}(\bar{g})$ for $n \leqslant -1$.

The mappings Λ_n appear naturally in the classification of left actions of SU(2) on SU(2) bundles over S^2. They guarantee the triviality of the bundles $\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathrm{SU}(2)$ (Harnad *et al* 1980).

Λ_n can be projected to the fibre bundle homomorphism $f_n: L_n \to L_1$, such that $f_n(gZ_n) = \Lambda_n(g)$, and further to a mapping $\Phi_n: S^2 \to S^2$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} & & \Lambda_n & & \\ \mathrm{SU}(2) & \xrightarrow{p_n} & \mathrm{SU}(2)/Z_n & \xrightarrow{f_n} & \mathrm{SU}(2) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_1} \\ S^2 & \xrightarrow{\mathrm{id}} & S^2 & \xrightarrow{\Phi_n} & S^2 \end{array} \tag{7}$$

L_n is the bundle induced by Φ_n; however, the connection ω_n is not that one induced by Φ_n from ω_1. All the mappings in the diagram are of class C^∞. Λ_n belongs to the homotopy class $[n^2]$ in $\pi_3(\mathrm{SU}(2))$ whereas Φ_n represents the element $[n]$ of $\pi_2(S^2)$. In the coordinates θ, ϕ

$$\Phi_n(\theta, \phi) = \sin\theta_n \cos n\phi\, e_1 + \sin\theta_n \sin n\phi\, e_2 + \cos\theta_n\, e_3, \tag{8}$$

where

$$\tan(\theta_n/2) = \tan^{|n|}(\theta/2). \tag{9}$$

The bundles L_n ($n=0$ included) exhaust all inequivalent U(1) bundles over S^2. Each of them admits a (unique) left action of SU(2) commuting with (3) and projecting to the rotations

$$\pi(g') \to \pi(gg') = g\pi(g')g^{-1}, \qquad g \in \mathrm{SU}(2) \tag{10}$$

of S^2. These actions are given by

$$(\pi(g'), a) \to (\pi(gg'), a) \qquad \text{for } n=0, \tag{11a}$$

$$g'Z_n \to (gg')Z_n \qquad \text{for } n \neq 0. \tag{11b}$$

The connection form ω_n is invariant under (11) with the appropriate index n, that corresponds to the spherical symmetry (up to gauge transformations) of the potentials (6).

Now we consider the Dirac monopoles from the point of view of the SU(2) Yang–Mills theory. Wu and Yang (1975) first noted that if the gauge group of electromagnetism is extended to SU(2) then potentials of the monopole with $n=1$ are free from string singularities in an appropriate gauge. There were a few attempts (Quiros and Rodriguez 1983, Bais 1976) to extend this result for $|n|>1$, however, in our opinion they are not satisfactory.

There is only one (up to equivalences) SU(2) principal bundle over S^2, namely the trivial one $S^2 \times \mathrm{SU}(2)$. The bundles L_n can be considered as its reductions. The corresponding embeddings I_n: $L_n \to S^2 \times \mathrm{SU}(2)$ are given by

$$I_0(x, a) = (x, a), \qquad I_n(gZ_n) = (\pi(g), \Lambda_n(g)) \qquad \text{for } n \neq 0. \tag{12a, b}$$

ω_n defines uniquely a connection $\tilde{\omega}_n$ on $S^2 \times \mathrm{SU}(2)$ such that $\omega_n = I_n^* \tilde{\omega}_n$. Due to the triviality of $S^2 \times \mathrm{SU}(2)$, $\tilde{\omega}_n$ necessarily takes the form

$$\tilde{\omega}_n(x, h) = h^{-1}\tilde{A}_n(x)h + h^{-1}\,\mathrm{d}h,$$

where $\tilde{A}_n$ is a 1-form of class C^∞ on S^2. $\tilde{A}_0 = 0$ and $\tilde{A}_n$ for $n \neq 0$ can be computed by the use of (5), hence

$$(\pi^*\tilde{A}_n)(g) = \Lambda_n(g) n e_3 \theta^3 \Lambda_n(g)^{-1} + \Lambda_n(g)\,\mathrm{d}\Lambda_n(g)^{-1}, \tag{13}$$

and further

$$\tilde{A}_n(\theta, \phi) = (\sin n\phi\, \mathrm{d}\theta_n + n \sin\theta_n \cos\theta \cos n\phi\, \mathrm{d}\phi)e_1 \\ + (-\cos n\phi\, \mathrm{d}\theta_n + n\sin\theta_n \cos\theta \sin n\phi\, \mathrm{d}\phi)e_2 + n(\cos\theta_n \cos\theta - 1)\,\mathrm{d}\phi\, e_3, \tag{14}$$

where θ_n is given by (9).

From the viewpoint of the Yang–Mills theory on the Minkowski space, the forms $\tilde{A}_n$ represent the potentials of Dirac monopoles in a no-string gauge. They are singular at $r=0$ only. To get physical quantities, $\tilde{A}_n$ can be divided by the coupling constant related to the gauge group SU(2).

The procedure leading to $\tilde{A}_n$ can be described without the notion of fibre bundles. To do this we first set $\kappa = n/2$ in (1) and multiply the result by $(-i\sigma_3)$. In this way we get the potential form A_n^+ (defined for $\theta \neq 0$) in the framework of the SU(2) gauge theory. Next we perform the gauge transformation $A_n^+ \to h_n^{-1} A_n^+ h_n + h_n^{-1} \mathrm{d}h_n$, where

$$h_n = [\sin^{2n}(\theta/2) + \cos^{2n}(\theta/2)]^{-1/2} \begin{pmatrix} \cos^n(\theta/2)\exp(\mathrm{i}n\phi), & \sin^n(\theta/2) \\ -\sin^n(\theta/2), & \cos^n(\theta/2)\exp(-\mathrm{i}n\phi) \end{pmatrix}.$$

The resulting expression coincides with $\tilde{A}_n$ and is extendable to whole S^2.

It follows from (7) and (13) that

$$\mathrm{d}\Phi_n + [\tilde{A}_n, \Phi_n] = 0,$$

hence Φ_n defines a covariantly constant Higgs field (in the adjoint representation) on S^2. Since the winding number of Φ_n is equal to n, so $\tilde{A}_n$ and Φ_n can be asymptotic forms of the gauge field and the Higgs field, respectively, appearing in the construction of non-Abelian monopoles (see Goddard and Olive (1978) and O'Raifeartaigh and Rouhani (1981) for a review).

The forms $\tilde{A}_{\pm 1}$ are equivalent to the potentials found by Wu and Yang (1975). For $n \neq 0, \pm 1$ $\tilde{A}_n$ and Φ_n are different from the expressions considered by Quiros and Rodriguez (1983) and Bais (1976), which correspond to (8) and (14) with θ_n replaced by θ. The potentials obtained by these authors (for $n \neq 0$, ± 1) have singularities at $\theta = 0$ and π, whereas the Higgs fields are not differentiable at these points. Thus they cannot be asymptotic quantities for non-Abelian monopoles unless singular gauges are used.

Considering possible left actions of SU(2) on $S^2 \times \mathrm{SU}(2)$ there are infinitely many non-equivalent actions, which commute with the action of the structure group and project to the rotations (10) of S^2. All of them can be deduced from (11) and (12) and are given by

$$(\pi(g'), h) \to (\pi(gg'), \Lambda_n(gg')\Lambda_n(g')^{-1}h), \qquad g \in \mathrm{SU}(2),$$

where $\Lambda_0(g) = g$. The connection $\tilde{\omega}_n$ is invariant under the action with the index n, hence $\tilde{A}_n$ is invariant under rotations up to gauge transformations.

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